M332 GE

THE OPEN UNIVERSITY

Mathematics: A Third Level Course

A GUIDE TO THE EXAMINATION: COMPLEX ANALYSIS

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PREPARING FOR THE EXAMINATION

The real key to success in any examination is to work steadily and systematically during the course! Plan your work for each unit, and keep in a stout file and in proper order all your notes on units, all the examples and exercises you solved, and all your TMA solutions. This will make it easy for you to revise, and will save you a great deal of time and nervous energy!

Make good use of your Course Tutor. Make sure that you read all the comments your tutor makes on your work, *especially* the advice on which parts of the course you should spend some extra time mastering. Sometime before the exam, have a word with your tutor at a Tutorial or Day School (or by telephone or letter) to discuss your progress during the year and your plans to revise the course.

To help you with your revision, we have prepared a Revision Booklet which consists of problems covering topics in *Units 1–11*. Try to go through this booklet with enough time in hand to do any extra revision you find is necessary!

We have also sent you a Specimen Examination Paper and Specimen Solutions. This will give you an idea of the sort of questions to expect, and also the sort of answers to write. Go through it carefully at some appropriate time during your revision. You may find it useful to treat the Specimen as a timed paper. The Specimen Paper highlights important features of examination technique. Notice that if few marks are awarded for some part, you will be expected to write less and spend less time on it than if a lot of marks were assigned to it. Except where specifically told that no explanation is required, if you invoke a major theorem you should mention it specifically and check that it does apply. When the Examination and Assessment Board makes up your examination paper, it will follow the general pattern of the Specimen Paper, and will choose the definitions, statements of theorems, proofs of theorems, and examples using particular techniques from the list which appears later in this Guide.

Finally, make sure you schedule your revision over as long a period as possible so that you are not rushed as the examination day approaches.

ADVICE FOR THE DAY OF THE EXAMINATION

Before the Examination

Make sure you are fully prepared. *Check* the time and place of your examination centre so that you arrive in plenty of time and do not arrive tired, flustered or late. Bring along several pens and pencils.

In this course you cannot take into the exam room any of the course materials.

Read the Paper Carefully

This sounds very obvious!—but it is so easy under the strain of an examination to plunge in, especially when you are surrounded by other candidates who are all scribbling away at a furious rate. Take your time—spend at least five minutes reading the paper and noting which questions you definitely can do, which ones (if any!) you definitely cannot, and which ones you might be able to tackle.

If you are careful with your choice of first question, and get it right, this gives you some confidence and gets rid of any exam nerves you may have had. Most students do best by first "knocking off" some questions from Part I of the paper; you are sure to find something in Part I that you know all about!

Do Not Panic

This is very easy to say, but it is *essential* to keep a cool head throughout the examination.

If, after about ten or fifteen minutes, you find that a question is leading you nowhere or that you are getting bogged down in very heavy manipulation, start a different question. It often happens that when you come back to the original question you have a new idea, or spot a mistake in what you had written before. *Do not* battle on with a question if you are getting nowhere with it!

Write Clearly

In each question, there are marks assigned to both the methods you use and to the accuracy of your calculations. So the more clearly you write and the better you lay out your solutions, the easier it will be for the examiner to award you the appropriate marks.

If the examiner is unable to decipher what you have written, he will eventually have to ignore it and give you no marks for it—even if it is, in fact, correct.

Even if your final answer is wrong, you may well get partial marks for some parts of a question you have attempted, if the examiner can read what you have done.

Explain your Working

If you miss out a step in a proof or a calculation, the examiner has no way of knowing whether this was through ignorance or whether you really did know about it. You cannot expect him to give you the benefit of the doubt; so, unless you are very pressed for time, *always* include an explanation of what you are doing.

Check your Answers

Towards the end of the examination, you should spend some time checking your answers. Five minutes spent checking may gain you more marks than five minutes on a new question.

Handing In your Answers

Make sure you hand in all your solutions, following the instructions at the front of the paper.

After the Examination

Forget all about it! It is too late to do anything about it now, so do not worry about what you should have written. And do not be over-impressed by someone else's claim about how well he *thinks* he has done!

FORMAT OF THE EXAMINATION PAPER

The paper will consist of three parts.

Part I will contain 10 questions, and carries 50% of the total marks.

Each question is fairly short, and tests one or two basic concepts and techniques from *Units 1–11*. They cover most of the important topics in these units.

Part II will contain 6 questions. You should answer at most TWO of these questions; each is worth 17% of the total marks, so that Part II is worth 34% all told. These questions also cover material from *Units 1–11*, but they are longer and more searching than questions in Part I.

Part III will contain 2 questions, one on *Unit 13* and one on *Unit 14*. You should answer at most ONE of these questions. Part III is worth 16% of the total marks.

The rubric and form of your examination paper will be similar to those of the Specimen Paper. We suggest that you examine this carefully so as to avoid wasting valuable time during the actual examination.

The Bookmarks have already listed what you should know from each unit for the examination, but for your convenience we reprint here the complete list for the course.

"Definitions" Part of a question may say something like "define an X" or "say what you mean by X".

"Statements of results" Part of a question may say something like "state the such-and-such Theorem" or "state conditions under which such-and-such a Theorem applies".

"Proofs" Some questions will ask you to "prove X's Theorem" or to "outline the proof of such-and-such a Theorem". If asked for a complete proof, give it; if asked for an outline, just give an outline.

Sometimes you may be asked to state *clearly* any preliminary result that you require to use in your proof.

"Techniques" These list all the different types of problems you may be asked to do in the examination. Not all of them will be asked each year, so do not worry!

WHAT YOU SHOULD KNOW FOR THE EXAMINATION

UNIT 1

DEFINITIONS		STATEMENTS OF RESULTS		PROOFS		TECHNIQUES
p.		p.		p.		
		30	de Moivre's Theorem			Find real part, imaginary part, conjugate, modulus, polar form, argument, principal value of the argument of a complex number. Find nth roots of a complex number. Sketch curves and sets in $\mathbb{C}$.
		39	Nested Rectangles Theorem			
		42	Triangle Inequality $\lvert z+w\rvert \leqslant \lvert z\rvert + \lvert w\rvert$	42	Triangle Inequality	
		42	"Cunning form of Triangle Inequality" $\lvert z-w\rvert \geqslant \lvert\lvert z\rvert - \lvert w\rvert\rvert$			
43	Cluster point	43	Bolzano–Weierstrass Theorem			

UNIT 2

54	Inverse image					Find images and inverse images of sets under a given function. Determine unions, intersections, complements, closures, boundaries of sets. Find if a set is open, closed, compact, connected, region, polygonally-connected.
59	f has a *limit* near a					
62	Interior point					
63	Open set					
70	Closed set					
70	Closure of set					
72	Boundary point					
73	f is *continuous* at a					
77	Open set is *disconnected*					
77	Open set is *connected*					
80	*Polygonally-connected* set					
81	Region					
89	Compact set					

UNIT 3

102	Differentiable					From definition, show that a given function cannot be differentiable.
102	Analytic					
102	Entire					Prove identities and solve equations involving trigonometric and hyperbolic functions.
120	Exponential function					
123	$\sin z$, $\cos z$					Determine values of $\operatorname{Log} z$ and z^{α}.
125	$\sinh z$, $\cosh z$					
130	Principal branch of logarithm					Use the Cauchy–Riemann equations to decide whether or not a function is differentiable.
131	Fundamental region for exp					
131	Branch of logarithm associated with fundamental region					
134	Principal branch of the αth power					
140	Cauchy–Riemann equations			140	$f = u + iv$ analytic $\Rightarrow$ Cauchy–Riemann equations	

UNIT 4

20	Arc	42	Fundamental Theorem for contour integrals			Evaluation of integral by parametrization.
20	Path of an arc					
21	Simple arc	47	Estimation Theorem	47	Estimation Theorem (*stating* preliminary result required)	Sketching paths of arcs.
22	Closed arc					
23	Simple-closed arc					Determine whether an arc is simple, closed, simple-closed, smooth.
23	Smooth arc					
25	Length of a smooth arc					Evaluation of contour integral by Fundamental Theorem.
27	Path-connected					
30	Integral of f along γ					Find upper bound for modulus of integral by Estimation Theorem.
35	Winding number					

UNIT 5

55	Star region	63	Cauchy–Goursat Theorem	63	Cauchy–Goursat Theorem	Determine if a region is a star region, convex region.
55	Convex region	68	Antiderivative Theorem			
67	Antiderivative	74	Cauchy's Formula	74	Cauchy's Formula (*stating* preliminary result required)	Evaluation of contour integral by Cauchy's Formula, Cauchy's Formula for derivatives.
		78	Cauchy's Formula for first derivative	78	Cauchy's Formula for first derivative	Evaluation of integrals by parts and by substitution.
		82	Cauchy's Formula for nth derivative			
		88	Morera's Theorem			
		89	Liouville's Theorem	89	Liouville's Theorem	
		90	Fundamental Theorem of Algebra	90	Fundamental Theorem of Algebra	

UNIT 6

117	Radius of convergence of a power series	129	Taylor's Theorem	116	$\sum a_n z^n$ converges at z_0 $\Rightarrow$ convergence in $\lvert z\rvert < \lvert z_0\rvert$	Finding radius of convergence of power series by ratio and comparison tests.
141	Zero of order n	143	Uniqueness Theorem for analytic functions	120	$f(z) = \sum a_r z^r$ in open disc D $\Rightarrow f$ analytic in D	Determine a Taylor series by various techniques.
				120	$f(z) = \sum a_r z^r$ in $\lvert z\rvert < k$ $\Rightarrow f'(z) = \sum r a_r z^{r-1}$	Determine order of a zero.

UNIT 8

9	Removable singularity		Behaviour of function near singularity:			Determine and classify the singularities in $\mathbb{C}$ of a given function.
9	Pole of order m					
9	Essential singularity	11	(i) Theorem 1			Determine Laurent series of a given function.
		15	(ii) Theorem 3			
		22	Theorem 4			
		23	(iii) Casorati–Weierstrass Theorem	23	Casorati–Weierstrass Theorem	Determine residues by Laurent expansions, Cover-up Rule.
		38	Laurent's Theorem (Theorem 7)			
		48	Effect of type of singularity on the Laurent coefficients (Theorem 9)	47	Brief outline of Laurent's Theorem	
50	Residue of f at α					

UNIT 9

		60	Local Maximum Principle	60	Local Maximum Principle (*stating* preliminary result required)	Determine maximum of modulus of function on given set.
		62	Maximum Principle			
71	Simply-connected	65	Minimum Principle			Determine whether a region is simply-connected.
76	Analytic logarithm of f on R	103	Residue Theorem	76	Existence of analytic logarithm of non-zero analytic function on simply-connected region	Use of Cauchy's Theorem and Formulae in simply-connected regions.
						Determine winding numbers in various situations.
						Evaluation of integrals by Residue Theorem.

UNIT 10

			Statements of formulae and conditions for integrals of type $\int_{-\infty}^{\infty} \frac{p(t)}{q(t)} \begin{Bmatrix}\cos\\ \sin\end{Bmatrix} \beta(t)\,dt$:			Various techniques for evaluating residues. Evaluation of integrals by Residue Theorem. Evaluation of trig. integrals over $[0, 2\pi]$. Evaluation of integrals $\int_{-\infty}^{\infty} \frac{p(t)}{q(t)} \begin{Bmatrix}\cos\\ \sin\end{Bmatrix} \beta(t)\,\mathrm{d}t$.
		34	Theorem 5			
		36	Theorem 7			
		38	Theorem 8(a)			
		39	Theorem 8(b)			
		46	Theorem 9			
		47	Contours on which $\lvert\cot \pi z\rvert \leqslant K$	47	Contours on which $\lvert\cot \pi z\rvert \leqslant K$	Summing series $\sum_{1}^{\infty} \phi(n)$, $\sum_{1}^{\infty} (-1)^n \phi(n)$
53	Meromorphic	54	Principle of the Argument	54	Principle of the Argument	

UNIT 11

63	Pointwise convergence of $\{f_n\}$					Use of Theorem 3 and Weierstrass M-test to prove a given function is analytic.
64	Uniform convergence of $\{f_n\}$	68	Weierstrass M-test	65	$f_n \to f$ uniformly $\Rightarrow \int f_n \to \int f$	
82	Function element					Use of Rouché's Theorem.
82	Direct analytic continuation	82	Uniqueness of direct analytic continuation			Evaluation of integrals using Residue Theorem for Complete Analytic Functions (pp. 105–7).
85	Chain of function elements					
85	Complete chain	89	Monodromy Theorem			Analytic continuation by power series.
93	Description of Riemann surface for "log", and explanation of connections of "sheets" and of their inverse images					
116	Barrier point	102	Rouché's Theorem	102	Rouché's Theorem (*stating* preliminary result required)	
The examination will contain *no* questions on: { differentiation under the integral sign, gamma function, Schwarz reflection principle.						

UNIT 13

9	Conformal transformation	11	Open Mapping Theorem	8	Analytic with non-zero derivative at $\alpha \Rightarrow$ conformal (Theorem 1)	Use fact that bilinear transformations map circles/lines onto circles/lines (determined by images of 3 points).
22	Bilinear transformation	32, 33	A bilinear transformation mapping $\lvert z\rvert < 1$ onto itself is necessarily of form $z \to \frac{az+b}{\bar{b}z+\bar{a}}$ $(\lvert b\rvert < \lvert a\rvert)$	32	$z \to \frac{z-\alpha}{1-\bar{\alpha}z}$ maps $\lvert z\rvert < 1$ onto itself, with $\alpha \to 0$	Determine analytic equivalences of a region with another, via composition of bilinear transformations.
				37	Bilinear transformation is conformal	$z \to z^\alpha$, $z \to \exp z$, $z \to \sin z$, $z \to \cos z$ (cf. Examples 1–6, p. 62).
47	Continuous extension of f to $\hat{\mathbb{C}}$	60	Riemann Mapping Theorem	47	$\hat{f}: \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ is 1–1 $\Rightarrow f$ is bilinear transformation	
There will be no questions on non-Euclidean geometry.						

UNIT 14

19 19 45	Laplace transform Inversion formula Asymptotic expansion of f for large t	 23 47	 State conditions under which inversion formula holds Watson's Lemma	 20 20 29 34 	Laplace transforms of: $f(t) = e^{at}$ $f'(t)$ $f''(t)$ Outline the main steps in using the theory of residues to invert a Laplace transform, stating *carefully* the two main theorems used (Theorem 1, p. 23; Theorem 6, p. 31)	Solve a second-order ordinary differential equation with constant coefficients $af'' + bf' + cf = g$ for some g, by taking Laplace transforms of both sides, solving for $F = \mathscr{L}[f]$, and then using the inversion formula to determine f. Determine an asymptotic expansion for $F = \mathscr{L}[f]$ by Watson's Lemma.
There will be no questions on the Method of Steepest Descent.						

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